Proof for LP strong duality:

Here, consider min c^Tx, s.t $Ax=b$, $x\geq 0$, $x\in\mathbb{R}^n$, $A\in\mathbb{R}^{m\times n}$, $b\in\mathbb{R}^m$

i.e. we only consider the standard form LP since any kind of LP can be transformed to it.

(Assume $C, D\subseteq\mathbb{R}^n$).

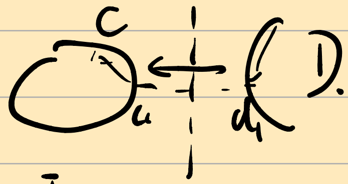
Lemma 1: (Seperation thm): For two closed and convex set, denoted by C and D, if one is bounded, then we have: $\exists\, d\neq 0\in\mathbb{R}^n$ and $r\in\mathbb{R}$, s.t $d^Tc<r<d^Te$ for $\forall c\in C, d\in D$. if $C\cap D=\emptyset$.

Lemma's pf: W.L.O.G assume C is bounded.
It's easy to verify that: $\exists\, c_1\in C, d_1\in D$, s.t $d(C,D)=\|c_1-d_1\|_2$.

Here, $d(C,D)=\inf_{c\in C, d\in D}\|c-d\|_2=\inf_{c\in C}\rho(c,D)$

(Can be down by choosing: $d(c_n,d_n)\downarrow d(C,D)$ within $\frac{1}{n}$ respectively
Or just leverage $\rho(c,D)$ is cts and C is cpt, D is cls).

Then, let $d=c_1-d_1$, $r=\frac{c_1^2-d_1^2}{2}$



Since $\begin{cases} d_1=\arg\min_{D}\frac{1}{2}\|c_1-d\|^2\\ c_1=\arg\min_{C}\frac{1}{2}\|c-d_1\|^2\end{cases}$, we get $\begin{cases}(c_1-d_1)^T(d_1-d)\geq 0, \text{ any } d\in D\\ (c_1-d_1)^T(c-c_1)\geq 0, \text{ any } c\in C\end{cases}$

Hence, $\begin{cases}(c_1-d_1)^Td_1\geq (c_1-d_1)^Td, \text{ any } d\in D\\ (c_1-d_1)^Tc\geq (c_1-d_1)^Tc_1, \text{ any } c\in C.\end{cases}$

Hence, $(c_1-d_1)^Tc\geq (c_1-d_1)^Tc_1>(c_1-d_1)^Td_1\geq (c_1-d_1)^Td$, any $c,d\in C,D$.

It's easily to verify that: $(c_1-d_1)^Tc_1>(c_1-d_1)^T\frac{c_1+d_1}{2}>(c_1-d_1)^Td_1$.

// In $\mathbb{R}^n$. (implies without bdd, even if $d(C, D) > 0$, we can't get the strict result).

• Corollary: For any closed convex set C, any point $d \notin C$, we have: $\exists\, e \neq 0$, s.t $e^T d > e^T c$ for any $c \in C$.

• Corollary: For any cvx set $C \neq \emptyset \subseteq \mathbb{R}^n$, $z \in \text{bdry}(C)$, $\exists\, a \neq 0 \in \mathbb{R}^n$. s.t $a^T z \geq a^T x$, $\forall x \in C$.

(Just using $cl(C)$ is closed cvx set. Since $z \in \text{bdry}(C)$, $\exists\, z_i \in (cl(C))^c$ with $\|a_i\| = 1$, s.t $a_i^T z_i > a_i^T x$, $\forall x \in cl(C)$ and $z_i \to z$.
$\exists \{a_{i_k}\}_{k=1}^{\infty} \subset \{a_i\}_{i=1}^{\infty}$ converging to a, then we get:

$a^T z \geq a^T x$, $\forall x \in C$.).

• Corollary: For any two cvx set C, D, if $C \cap D = \emptyset$, then $\exists\, a \neq 0$, s.t $a^T c \leq a^T d$, $\forall c \in C$, $\forall d \in D$.

(Considering $C - D$, $0 \notin C - D$. consider $\begin{cases} 0 \in cl(C-D) \\ 0 \notin cl(C-D) \end{cases}$)

Now, we can propose:

Lemma 2: (Farkas Strong Alternative).
$\begin{cases} ① \ \{x \mid Ax = b,\ x \geq 0\}. \\ ② \ \{y \mid A^T y \leq 0,\ b^T y > 0\} \end{cases}$, exactly one of ①, ② is empty

Proof: If ① is nonempty, ② must be empty.

If ① is empty, Let $A = (a_1, \cdots, a_n)$, then $b \notin \text{cone}(a_1, \cdots, a_n)$ which is closed. then, $\exists\, y \neq 0$, s.t $y^T b > y^T c$, $c \in \text{cone}(a_1, \cdots, a_n)$.

Since $0 \in \text{cone}(a_1, \cdots, a_n)$, $y^Tb > 0$.
And $y^Ta_i \le 0$. $i = 1 \sim n$. (Otherwise. $y^Tc \to \infty$ for some $c \in \text{cone}(a_1, \cdots, a_n)$)
$\Rightarrow y^TA \le 0 \ldots$

Lemma 3:
① $\exists x$, $Ax \le b$
② $\exists y \geq 0$, s.t $y^TA = 0$, $y^Tb < 0$. $\}$ exactly one of them holds

Proof: If ① holds, then ② can't holds.

If ① doesn't hold. Rewrite it in standard form:

$\{(x^+, x^-, s) \mid (A, -A, I)\begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} = b, x^+, x^-, s \geq 0\}$ is empty.

Hence, $\exists y \neq 0 \in \mathbb{R}^n$, s.t $\begin{pmatrix} A^T \\ -A^T \\ I \end{pmatrix} y \geq 0$, $b^Ty < 0 \Rightarrow \underset{A^Ty \geq 0}{A^Ty = 0}$.

Now, we begin our prove

(P) $\min\ c^Tx$ s.t $Ax = b$, $x \geq 0$

(D) $\max\ b^T\lambda$ s.t $c \geq A^T\lambda$

Proof: If (P) isn't bdd, i.e. $p^* = -\infty$, then $d^* = -\infty$ as well.

If p^* is finite, we just need to prove $\begin{cases} c \geq A^T\lambda \\ b^T\lambda \geq p^* \end{cases}$ is feasible, i.e.

$\begin{pmatrix} A^T \\ -b^T \end{pmatrix} \lambda \le \begin{pmatrix} c \\ -p^* \end{pmatrix}$ is feasible.

If not. $\exists (y_1, y_2) \geq 0$, s.t $Ay_1 - by_2 = 0$, $y_1^Tc - y_2p^* < 0$

① If $y_2 = 0$, then. $Ay_1 = 0$, $y_1^T c < 0$, $y_1 \geq 0$ $\Rightarrow$ Contradiction.

② If $y_2 \neq 0$, then $A\left(\frac{y_1}{y_2}\right) = b$, $c^T \frac{y_1}{y_2} < p^* \Rightarrow$ Contradiction. $\square$